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J. Math. Anal. Appl. 280 (2003) 232–240

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On existence of singular solutions

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Received 29 April 2002

Submitted by J.S.W. Wong

Abstract

In the paper sufficient conditions are given under which the differential equation $y^{(n)} = f(t, y, \dots, y^{(n-2)})g(y^{(n-1)})$ has a singular solution $y : [T, \tau) \to R$, $\tau < \infty$ fulfilling

$$\lim_{t\to\tau} \left|y^{(j)}(t)\right| = \infty, \quad j = 0, 1, \dots, n-1.$$

Consider the n-order differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(n-2)})g(y^{(n-1)}), \tag{1}$$

where $n \geqslant 2$, $f \in C^0(R_+ \times R^{n-1})$, $g \in C^0(R)$, $R_+ = [0, \infty)$, $R = (-\infty, \infty)$.

A solution y defined on $[T, \tau) \subset R_+$ is called singular if $\tau < \infty$ and y cannot be defined for $t = \tau$. Note that in this case $\limsup_{t\to\tau} |y^{(n-1)}| = \infty$. A singular solution y is called nonoscillatory if there exists a left neighbourhood of τ in which $y \neq 0$.

The problem of the nonexistence of singular solutions of (1) is solved by Wintner's theorem (see [8] or [11]); the main assumption is

$$\left|f(t, x_1, \dots, x_{n-1})g(x_n)\right| \leqslant r(t)\omega\left(\sum_{i=1}^{n} |x_i|\right) \quad \text{on } R_+ \times R^n,$$

where $\omega \in C^0(R_+)$ is nondecreasing, $\omega(x) > 0$ for $x > 0$, $\int_1^\infty (1/\omega(x))\,dx = \infty$ and $r \in C^0(R_+)$.

For the second order Emden–Fowler equation

$$y'' = r(t)|y|^\lambda \operatorname{sgn} y, \quad \lambda > 0, \tag{2}$$

where $r \leqslant 0$ on R_+ the nonexistence result is extended in [6,10].

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0022-247X/03/$ – see front matter
doi:10.1016/S0022-247X(02)00654-6

As concern the existence of singular solutions of (1), fulfilling Cauchy initial conditions, the first results for (2) are obtained in [9] and they are generalized for (1) and its special cases, e.g., in [3,5,11]. Note that if $r(t) \leqslant 0$ on R_+, there exists no theorem for the existence of singular solutions of (2), but an example of a function $r < 0$ is given in [4] for which (2) has a singular solution.

In this paper another problem will be studied—we will seek for singular solutions with prescribed asymptotics in the right-hand side point of the definition interval.

More precisely, the existence problem of a singular solution y of (1) fulfilling

$$\begin{aligned}
&\tau \in (0, \infty), \qquad l \in \{0, 1, \dots, n-2\}, \qquad c_i \in R, \quad i = 0, 1, \dots, l, \\
&\lim_{t \to \tau_-} y^{(i)}(t) = c_i \quad \text{for } i = 0, 1, \dots, l, \\
&\lim_{t \to \tau_-} \left|y^{(j)}(t)\right| = \infty \quad \text{for } j = l+1, \dots, n-1,
\end{aligned} \tag{3}$$

has been studied where y is defined in a left neighbourhood of τ. Sufficient (necessary) conditions for the existence of a solution y fulfilling (3) are given in [7] (case $n = 2$) and in [1,2] (case $n \geqslant 2$). Recall, that singular solutions fulfilling (3) are sometimes called black-hole solutions (see [7]).

In the present paper we extend our investigation to the case that is not obtained in (3). We will seek a singular solution y of (1) that fulfills

$$\tau \in (0, \infty), \qquad \lim_{t \to \tau_-} \left|y^{(i)}(t)\right| = \infty \quad \text{for } i = 0, 1, \dots, n-1. \tag{4}$$

Note that if y is a singular solution fulfilling (4) then

$$\operatorname{sgn} y^{(k)}(t) = \operatorname{sgn} y^{(s)}(t) \quad \text{for } k, s \in \{1, \dots, n-1\},$$

holds in a left neighbourhood of τ.

The following theorem gives us sufficient conditions for the nonexistence of solutions fulfilling (4).

Theorem 1. *Let $\tau \in (0, \infty)$ and I be a left neighbourhood of τ, $I \subset R_+$. Let $M \in (0, \infty)$ and $M_1 \in (0, \infty)$ be such that one of the following assumptions holds*:

(i) $$\begin{aligned}
&\alpha f(t, x_1, \dots, x_{n-1}) g(x_n) \leqslant 0 \quad \text{for } \alpha x_i \geqslant M, \\
&\quad i = 1, 2, \dots, n, \ \alpha \in \{-1, 1\}, \ t \in I;
\end{aligned}$$

(ii) $$\lambda > 1 + \frac{1}{n-1}$$

and

$$\begin{aligned}
&\alpha f(t, x_1, \dots, x_{n-1}) g(x_n) \geqslant M_1 |x_n|^{\lambda} \quad \text{for } \alpha x_i \geqslant M, \\
&\quad i = 1, \dots, n, \ \alpha \in \{-1, 1\}, \ t \in I.
\end{aligned}$$

Then there exists no solution y of (1) *fulfilling* (4).

Proof. Let y be a solution of (1) defined in a left neighbourhood $J \subset I$ of τ such that $\lim_{t\to\tau_-} y^{(j)}(t) = \infty$, $j = 0, 1, \dots, n-1$ (the case $\lim_{t\to\tau_-} y^{(j)}(t) = -\infty$, $j = 0, 1, \dots, n-1$ can be investigated similarly). Put $\alpha = 1$. It is clear that there exists $\tau_1 \in J$ such that

$$y^{(j)}(t) \geqslant M \quad \text{for } t \in J_1 = [\tau_1, \tau),\ i = 0, 1, \dots, n-1. \tag{5}$$

Let (i) be valid. Then (1) and (5) yield $y^{(n)}(t) \leqslant 0$ on J_1 that contradicts $\lim_{t\to\tau_-} y^{(n-1)} = \infty$ and $\tau < \infty$.

Let (ii) be valid. Put $\lambda_1 = 1/(\lambda - 1)$, hence $n - 1 - \lambda_1 > 0$. According to (1), (5) and the assumption (ii)

$$y^{(n)}(t) \geqslant M_1\big[y^{(n-1)}(t)\big]^{\lambda}, \quad t \in J_1,$$

and the integration on $[t, \tau) \subset J_1$ yields

$$y^{(n-1)}(t) \leqslant \big[(\lambda - 1)M_1(\tau - t)\big]^{-\lambda_1}, \quad t \in J_1.$$

Hence, Taylor series theorem yields

$$\infty = y(\tau) = \sum_{i=0}^{n-1} \frac{y^{(i)}(\tau_1)}{i!}(\tau - \tau_1)^i + \int_{\tau_1}^{\tau} \frac{(\tau - s)^{n-2}}{(n-2)!} y^{(n-1)}(s)\, ds$$
$$\leqslant M_4 + M_5 \int_{\tau_1}^{\tau} (\tau - s)^{n-2-\lambda_1}\, ds < \infty,$$

as $n - 2 - \lambda_1 > -1$; here

$$M_4 = \sum_{i=0}^{n-1} \frac{y^{(i)}(\tau_1)}{i!}(\tau - \tau_1)^i, \qquad M_5 = \frac{[(\lambda - 1)M_1]^{-\lambda_1}}{(n-2)!}.$$

The contradiction proves that y does not fulfill (4). □

To prove the existence of a singular solution fulfilling (4) we need the following lemma.

Lemma 1. *Let $[a, b] \subset R_+$, $\Phi \in C^0[a, b]$ and $\tilde{f} \in C^0([a, b] \times R^n)$ be such that*

$$\tilde{f}(t, x_1, \dots, x_n) \leqslant \Phi(t), \quad t \in [a, b],\ x_i \in R,\ i = 1, 2, \dots, n.$$

Then for arbitrary $c_i \in R$, $i = 0, 1, \dots, n-1$ the equation

$$u^{(n)} = \tilde{f}(t, u, \dots, u^{(n-1)})$$

has at least one solution fulfilling the boundary value conditions

$$u^{(i)}(a) = c_i, \quad i = 0, 1, \dots, n-2, \qquad u(b) = c_{n-1}.$$

Proof. It follows, e.g., from [11, Lemma 10.1], as the homogeneous problem $u^{(n)} = 0$, $u^{(i)}(a) = 0$, $i = 0, 1, \dots, n-2$, $u(b) = 0$ has the trivial solution only. □

The following theorem gives us a sufficient condition under which singular solutions with (4) exist.

Theorem 2. *Let $\tau \in (0,\infty)$, $I = [\bar{\tau}, \tau] \subset R_+$, $\alpha \in \{-1, 1\}$, $M \in (0,\infty)$ and $M_1 \in (0,\infty)$ be such that*

$$g(x) \geqslant |x|^\lambda \quad \text{for } \alpha x \geqslant M \tag{6}$$

and

$$\alpha f(t, x_1, \ldots, x_{n-1}) \geqslant M_1 \quad \text{for } \alpha x_i \geqslant M,\ i = 1, 2, \ldots, n-1,\ t \in I. \tag{7}$$

Let

$$1 < \lambda \leqslant 1 + \frac{1}{n-1}. \tag{8}$$

Then there exists a solution y of (1) *defined in a left neighbourhood of $t = \tau$ that fulfills* (4).

Proof. We prove the statement for $\alpha = 1$. For $\alpha = -1$ the proof is similar. Let $\lambda < 1 + 1/(n-1)$ and let

$$M_2 = \max\bigl\{f(t, x_1, \ldots, x_{n-1})\colon\ t \in I,\ M \leqslant x_i \leqslant 3Me^\tau,\ i = 1, 2, \ldots, n-1\bigr\},$$
$$\lambda_1 = \frac{1}{\lambda - 1}, \qquad M_3 = \bigl[M_1(\lambda - 1)\bigr]^{-\lambda_1} + 2M\tau^{\lambda_1}, \qquad k_0 > (n+1)M. \tag{9}$$

Hence, according to (8)

$$n - 1 - \lambda_1 < 0. \tag{10}$$

Let $T \in I$, $T < \tau$ be such that

$$\tau - T \leqslant 1, \qquad \int_M^{2M} \frac{ds}{g(s)} > M_2(\tau - T), \tag{11}$$

and denote $J = [T, \tau)$.

Consider an auxilliary two-point boundary-value problem: $k \in \{k_0, k_0 + 1, \ldots\}$,

$$y^{(n)} = f\bigl(t, \Phi_0(t, y), \ldots, \Phi_0(t, y^{(n-2)})\bigr) g\bigl(\Phi(t, y^{(n-1)})\bigr),$$
$$y^{(i)}(T) = M, \quad i = 0, 1, \ldots, n-2, \qquad \lim_{t \to \tau_-} y(t) = k, \tag{12}$$

where

$$\varphi(t) = M(n-1) + M_3[\lambda_1 - n + 1]^{-1}(\tau - t)^{-\lambda_1} \quad \text{if } n - 1 - \lambda_1 < 0,$$
$$\varphi(t) = M(n-1) + M_3(\tau - t)^{-1} \quad \text{if } n - 1 - \lambda_1 = 0,$$
$$\Phi(t, s) = \begin{cases} s & \text{for } M \leqslant s \leqslant M_3(\tau - t)^{-\lambda_1}, \\ M_3(\tau - t)^{-\lambda_1} & \text{for } s > M_3(\tau - t)^{-\lambda_1}, \\ M & \text{for } s < M, \end{cases}$$
$$\Phi_0(t, s) = \begin{cases} s & \text{for } M \leqslant s \leqslant \varphi(t), \\ \varphi(t) & \text{for } s > \varphi(t), \\ M & \text{for } s < M. \end{cases}$$

Note, that according to the definition of M_3, Φ is well-defined.

To prove the existence of a solution of (12), let us consider the sequence of boundary value problems

$$\begin{aligned}
&\bar{m} > \frac{1}{\tau - T}, \quad m \in \{\bar{m}, \bar{m}+1, \ldots\}, \quad \tau_m = \tau - \frac{1}{m}, \quad J_m = [T, \tau_m],\\
&Z^{(n)} = F(t, Z, Z', \ldots, Z^{(n-1)}),\\
&Z^{(i)}(T) = M, \quad i = 0, 1, \ldots, n-2, \qquad Z(\tau_m) = k, \qquad (13)
\end{aligned}$$

where

$$F(t, x_1, \ldots, x_n) = f\bigl(t, \Phi_0(t, x_1), \ldots, \Phi_0(t, x_{n-1})\bigr) g\bigl(\Phi(t, x_n)\bigr). \qquad (14)$$

As evidently $|F(t)|$ is bounded from above by a continuous function on J_m, Lemma 1 yields the existence of a solution Z_m of (13). It is clear that according to (6), (7) and (14)

$$Z_m^{(n)} > 0, \qquad Z_m^{(n-1)} \text{ is increasing on } J_m. \qquad (15)$$

Further, we estimate the derivatives of Z_m. First we prove that

$$M < Z_m^{(n-1)}(t), \quad t \in J_m, \; m = \bar{m}, \bar{m}+1, \ldots. \qquad (16)$$

For this, we prove that

$$Z_m^{(n-1)}(\tau_m) > 2M. \qquad (17)$$

Let, contrarily, $Z_m^{(n-1)}(\tau_m) \leqslant 2M$. Then the Taylor series theorem, (11) and (15) yield

$$\begin{aligned}
k_0 \leqslant k = Z_m(\tau_m) &= M \sum_{i=0}^{n-2} \frac{(\tau_m - T)^i}{i!} + \int_T^{\tau_m} \frac{(\tau_m - s)^{n-2}}{(n-2)!} Z_m^{n-1}(s)\, ds\\
&\leqslant M(n-1) + 2M = (n+1)M,
\end{aligned}$$

that contradicts (9). Hence (17) holds and if (16) is not valid there exist T_1 and T_2 such that $T \leqslant T_1 < T_2 < \tau_m$, $Z_m^{(n-1)}(T_1) = M$ and $Z_m^{(n-1)}(T_2) = 2M$. Then, according to (11), (15) and Taylor series theorem, we have

$$\begin{aligned}
M \leqslant Z_m^{(i)}(t) &= M \sum_{j=0}^{n-i-2} \frac{(t - T_1)^j}{j!} + \int_{T_1}^{t} \frac{(t - s)^{n-i-2}}{(n-i-2)!} Z_m^{(n-1)}(s)\, ds\\
&\leqslant M e^{\tau} + 2M \int_{T_1}^{t} e^{t-s}\, ds \leqslant 3M e^{\tau}, \quad t \in J_m, \; i = 0, 1, \ldots, n-2. \qquad (18)
\end{aligned}$$

It follows from the definition of M_3 that $2M \leqslant M_3(\tau - t)^{-\lambda_1}$ for $t \in [T_1, T_2]$ and as $y^{(n-1)}$ is increasing (see (15)) we have $g(\Phi(t, y^{(n-1)}(t))) = g(y^{(n-1)}(t))$, $t \in [T_1, T_2]$. From this and from (9) and (18) we have

$$Z_m^{(n)}(t) \leqslant M_2 g\bigl(Z_m^{(n-1)}(t)\bigr), \quad t \in [T_1, T_2],$$

and hence,

$$\int_M^{2M} \frac{ds}{g(s)} \leqslant M_2(T_2 - T_1) \leqslant M_2(\tau - T).$$

The contradiction with (11) proves that (16) holds.

Further, we prove that

$$Z_m^{(n-1)}(t) < M_3(\tau - t)^{-\lambda_1}, \quad t \in [T, \tau_m), \tag{19}$$

holds for large m, say $m \geqslant \bar{m}_0$. If (19) is not valid, then either

(i) there exists $t_1 \in [T, \tau_m)$ such that

$$Z_m^{(n-1)}(t_1) = M_3(\tau - t_1)^{-\lambda_1}, \qquad Z_m^{(n-1)}(\tau_m) \leqslant M_3(\tau - \tau_m)^{-\lambda_1}, \tag{20}$$

or

(ii) $$Z_m^{(n-1)}(t) > M_3(\tau - t)^{-\lambda_1}, \tag{21}$$

in a left neighbourhood of $t = \tau_m$.

Let (i) be valid. It follows from (7), (13)–(16) that

$$Z_m^{(n)}(t) \geqslant M_1\big(Z_m^{(n-1)}(t)\big)^{\lambda}, \quad t \in [t_1, \tau_m],$$

and the integration on $[t_1, \tau_m]$, (8) and (20) yield

$$\frac{\tau_m - t_1}{M_3^{\lambda-1}} \geqslant \big[Z_m^{(n-1)}(t_1)\big]^{1-\lambda} - \big[Z_m^{(n-1)}(\tau_m)\big]^{1-\lambda} \geqslant M_1(\lambda - 1)(\tau_m - t_1).$$

The contradiction with the definition of M_3 shows that case (i) is impossible.

Let (ii) hold and let $t_1 \in [T, \tau_m)$ be such that

$$Z_m^{(n-1)}(t) > M_3(\tau - t)^{-\lambda_1} \quad \text{on } [t_1, \tau_m).$$

Then the Taylor series theorem and (13) yield

$$\begin{aligned} k = Z_m(\tau_m) &\geqslant \int_{t_1}^{\tau_m} \frac{(\tau_m - s)^{n-2}}{(n-2)!} Z_m^{(n-1)}(s)\, ds \geqslant \frac{M_3}{(n-2)!} \int_{t_1}^{\tau_m} (\tau_m - s)^{n-2} (\tau - s)^{-\lambda}\, ds \\ &= \frac{-m M_3}{(n-1)!} \int_{t_1}^{\tau_m} (\tau - s)^{n-\lambda_1} \frac{d}{ds}\left(\left(1 - \frac{1}{m(\tau - s)}\right)^{n-1}\right) ds \\ &\geqslant \frac{M_3 m}{(n-1)!} A \left(1 - \frac{1}{m(\tau - t_1)}\right)^{n-1} \to \infty, \quad \text{for } m \to \infty, \end{aligned}$$

where

$$A = (\tau - t_1)^{n-\lambda_1} \quad \text{for } n - \lambda_1 < 0,$$
$$A = m^{\lambda_1 - n} \quad \text{for } 0 \leqslant n - \lambda_1 < 1,$$

as $n - 1 - \lambda_1 < 0$. Hence, (19) holds and the other derivatives of Z_m may be estimated.

It follows from the Taylor series theorem and from (11) that

$$M \leqslant Z_m^{(i)}(t) = M \sum_{j=0}^{n-i-2} \frac{(t-T)^j}{j!} + \int_T^t \frac{(t-s)^{n-i-2}}{(n-i-2)!} Z_m^{(n-1)}(s)\,ds$$
$$\leqslant M(n-1) + M_3 \int_T^t \frac{(\tau - s)^{n-i-2-\lambda_1}}{(n-i-2)!} \leqslant \varphi(t), \quad t \in J_m. \tag{22}$$

From this, from (16) and (19), $\{Z_m^{(j)}\}$, $j = 0, 1, \ldots, n-1$, $m = \bar{m}_0, \bar{m}_0 + 1, \ldots,$ are uniformly bounded with respect to j and m and hence, according to Arzela–Ascoli theorem (see [6, Lemma 10.2]), there exists a subsequence that converges uniformly to a solution y_k of (12). Moreover, the estimations (16), (19) and (22) hold also for y_k and hence $\Phi_0(t, y_k^{(i)}(t)) = y_k^{(i)}(t)$, $i = 0, 1, \ldots, n-2$, $\Phi(t, y_k^{(n-1)}(t)) = y_k^{(n-1)}(t)$, $t \in J$. As (16), (19) and (22) do not depend on k and i, Arzela–Ascoli theorem yields the existence of a subsequence of $\{y_k\}_{k_0}^{\infty}$ that converges uniformly to a solution y of (1) fulfilling $\lim_{t \to \tau_-} y(t) = \infty$ and (4) holds.

Let $\bar{\lambda} = 1 + 1/(n-1)$. Let $\{\lambda_s\}_1^{\infty}$ be a sequence such that $\lambda_s > 1$, $\lim_{s \to \infty} \lambda_s = \bar{\lambda}$. Note, that according to (9) and (11) T is independ on the parametr λ and hence, there exist a solution y_s of (1), (4) with $\lambda = \lambda_s$, $s \in \{1, 2, \ldots\}$ defined on $[T, \tau)$. Moreover, denote by $M_3(\lambda_s)$ the number M_3 from (9) for $\lambda = \lambda_s$. It is clear that

$$\overline{M}_3 = \sup_{s \in \{1,2,\ldots\}} M_3(\lambda_s) < \infty.$$

If M_3 is replaced by $\overline{M}_3$, the estimations (16), (19) and (22) are valid for $Z_m = y_s$, $s = 1, 2, \ldots,$ too, and moreover, they do not depend on s. Hence, there exists Φ such that

$$\left|y_s^{(i)}(t)\right| \leqslant \Phi(t), \quad t \in [T, \tau),\ i = 0, 1, \ldots, n-1,\ s = 1, 2, \ldots,$$

and according to Arzela–Ascoli theorem, there exists a subsequence of $\{y_s\}_1^{\infty}$ that converges uniformly to a solution of (1), fulfilling (4) with $\lambda = \bar{\lambda}$. □

Corollary 1. *Let $\tau \in (0, \infty)$, $\lambda > 1$, $I = [\bar{\tau}, \tau] \subset R_+$, $M \in (0, \infty)$ and $M_1 \in (0, \infty)$ be such that*

$$g(x) \geqslant |x|^{\lambda} \quad \textit{for } |x| \geqslant M, \tag{23}$$

and (7) holds for $\alpha = 1$ and for $\alpha = -1$. Then there exists a singular solution y of (1) fulfilling (4) if and only if

$$1 < \lambda \leqslant 1 + \frac{1}{n-1}.$$

Proof. It follows from Theorem 1(ii) and Theorem 2. □

Consider a special case of (1)

$$y^{(n)} = r(t)F(y)g(y^{(n-1)}), \tag{24}$$

where $\lambda > 0$, $r \in C^0(R_+)$, $F \in C^0(R)$, $g \in C^0(R)$ and let $M \in (0, \infty)$ and $\beta \in \{-1, 1\}$ be such that (23) and

$$\beta r(t) > 0 \quad \text{on } R_+, \qquad F(x)x > 0 \quad \text{for } x \neq 0$$

hold.

Theorem A [2]. *Let $c_0 \neq 0$. Then* (23) *has a singular solution y fulfilling* (3) *if*

$$1 + \frac{1}{n-l-1} < \lambda \leqslant 1 + \frac{1}{n-l-2} \quad \textit{for } l < n-2, \tag{25}$$

and

$$2 < \lambda \quad \textit{for } l = n-2.$$

If, moreover, a constant $M_1 > 1$ exists such that $g(x) \leqslant M_1|x|^\lambda$ for $|x| \geqslant M$ then (23) *has a singular solution y fulfilling* (3) *if and only if* (25) *is valid.*

Applying Corollary 1 to (24) we obtain

Corollary 2. *Let $\lambda > 1$, $\beta = 1$, $\tau \in (0, \infty)$ and $\liminf_{|x|\to\infty} |F(x)| > 0$. Then* (24) *has a singular solution y fulfilling* (4) *if and only if*

$$1 < \lambda < 1 + \frac{1}{n-1}. \tag{26}$$

Note, that the intervals (25) (for different l) and (26) are disjunct. Moreover, they altogether cover the interval $(1, \infty)$ in case $\beta = 1$ (the interval $(1 + 1/(n-1), \infty)$ in case $\beta = -1$).

As a singular solution y of (24) is nonoscillatory if, and only if $\lim_{t\to\tau_-} |y^{(n-1)}(t)| = \infty$ (see, e.g., [1]), the following result follows from Corollary 2 and Theorem 1(i).

Corollary 3. *Let* $\liminf_{|x|\to\infty} |F(x)| > 0$.

(i) *If $\beta = 1$ and $\lambda > 1$, then* (24) *has a nonoscillatory singular solution.*
(ii) *If $\beta = -1$, then* (24) *has a nonoscillatory singular solution if and only if $\lambda > 1 + 1/(n-1)$.*

Note, that the result of Corollary 3(i) is known, see, e.g., [11, Theorem 11].

The existence of a singular solution fulfilling (4) in an open problem in case $\lambda \leqslant 1$ and $\beta = 1$. The following example shows that this situation is possible.

Example. Let $\tau \in (0, \infty)$. Then $y = 1/(\tau - t)$, $t \in [0, \tau)$ is a singular solution of $y''' = 3|y''| \operatorname{sgn} y$, that fulfills (4).

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